Solution Sheet 2

1. (i) $q = 5, r = 15$ (ii) $q = 58, r = 15$ (iii) $q = -3, r = 7$ (iv) $q = -6, r = 3$.

   Importantly, the remainders are always **non-negative**.

2. ★ (i) $\gcd(97, 157) = 1 = 34 \times 97 - 21 \times 157$,

   (ii) $\gcd(527, 697) = 17 = 4 \times 527 - 3 \times 697$,

   (iii) $\gcd(2323, 1679) = 23 = 18 \times 1679 - 13 \times 2323$,

   (iv) $\gcd(4247, 2821) = 31 = 2 \times 4247 - 3 \times 2821$.

3. (i) $\gcd(44517, 15691) = 71 = 122 \times 15691 - 43 \times 44517$,

   (ii) $\gcd(173417, 159953) = 17 = 322 \times 159953 - 297 \times 173417$.

   See at the end of the sheet for the calculations.

4. (i) $5 \times 41 - 3 \times 68 = 1$, (ii) $5 \times 71 - 3 \times 118 = 1$,

   For $(3k + 2, 5k + 3)$ note that if you choose $k = 13$ you recover Part i while $k = 23$ gives Part ii. This observation might suggest considering the same linear combination seen in the answers to both parts, i.e.

   $$5 \times (3k + 2) - 3 \times (5k + 3) = 1.$$

   That we have an integer linear combination of $3k+2$ and $5k+3$ equalling to 1 is the definition of

   $$\gcd(3k + 2, 5k + 3) = 1,$$

   for all $k \in \mathbb{Z}$.

5.
   $$\begin{aligned} \gcd(a, c) &= 1 \Rightarrow \exists s, t \in \mathbb{Z} : sa + tc = 1, \\ \gcd(b, c) &= 1 \Rightarrow \exists p, q \in \mathbb{Z} : pb + qc = 1. \end{aligned}$$

   Rearrange as $sa = 1 - tc$ and $pb = 1 - qc$ and multiply together. After rearranging we get

   $$(sp)\, ab + (t + q - tqc)\, c = 1.$$

   That is, with $u = sp, v = t + q - tqc \in \mathbb{Z}$, we have $u\,(ab) + vc = 1$ which is the definition of $\gcd(ab, c) = 1$.

6. Always check your answers by substituting back in.

(i)★ By observation $m = -3, n = 2$ is a solution.

(ii) Without thinking we can use Euclid's algorithm to solve $2x+15y = \gcd(2, 15) = 1$, finding $2 \times -7 + 15 \times 1 = 1$. Multiply through by 4 to get the particular solution $m = -28, n = 4$.

Alternatively you could stare at $2m + 15n = 4$ for a minute to see that $m = 2$, $n = 0$ is a solution.

(iii)★ Euclid's Algorithm gives

$$\begin{aligned} 385 &= 12 \times 31 + 13 \\ 31 &= 2 \times 13 + 5 \\ 13 &= 2 \times 5 + 3 \\ 5 &= 3 + 2 \\ 3 &= 2 + 1 \end{aligned}$$

Working back we find that

$$1 = 12 \times 385 - 149 \times 31.$$

So a particular solution is $m = -149, n = 12$.

(iv) Euclid's Algorithm gives

$$\begin{aligned} 73 &= 41 + 32 \\ 41 &= 32 + 9 \\ 32 &= 3 \times 9 + 5 \\ 9 &= 5 + 4 \\ 5 &= 4 + 1. \end{aligned}$$

Working back we find that

$$1 = 9 \times 73 - 16 \times 41.$$

Multiply by 20 to get

$$20 = 180 \times 73 - 320 \times 41.$$

So a particular solution is $m = -320, n = 180$.

(v)★ With these small coefficients it is easy to see that $\gcd(93,81)=3$ which divides the right hand side of the Diophantine equation. Start by dividing through by $\gcd(93,81)=3$ to get $31m+27n=1$.

We quickly find by Euclid's Algorithm that $1=7\times 31-8\times 27$ (proving that $\gcd(31,27)=1$) so a particular solution is $m=7, n=-8$.

(vi) From Question 2(ii) we know that $\gcd(527,697)=17$ and $17\nmid 13$, hence the Diophantine Equation has no solutions.

(vii)★ Euclid's Algorithm gives

$$\begin{aligned} 533 &= 403+130, \\ 403 &= 3\times 130+13, \\ 130 &= 10\times 13. \end{aligned}$$

Hence $\gcd(533,403)=13$. Since $13|52$ the equation has solutions.

Working back we find that

$$13=4\times 403-3\times 533.$$

Multiply through by 4 to get

$$52=16\times 403-12\times 533,$$

giving a particular solution of $m=-12, n=16$.

3) i.

$$
\begin{aligned}
44517 &= 2 \times 15691 + 13135 \\
15691 &= 1 \times 13135 + 2556 \\
13135 &= 5 \times 2556 + 355 \\
2556 &= 7 \times 355 + 71 \\
355 &= 5 \times 71 + 0.
\end{aligned}
$$

Hence $\gcd(44517, 15691) = 71$, the last non-zero remainder.

Working back up

$$
\begin{aligned}
71 &= 2556 - 7 \times 355 \\
&= 2556 - 7 \times (13135 - 5 \times 2556) \\
&= 36 \times 2556 - 7 \times 13135 \\
&= 36 \times (15691 - 1 \times 13135) - 7 \times 13135 \\
&= 36 \times 15691 - 43 \times 13135 \\
&= 36 \times 15691 - 43 \times (44517 - 2 \times 15691) \\
&= 122 \times 15691 - 43 \times 44517.
\end{aligned}
$$

ii)

$$
\begin{aligned}
173417 &= 1 \times 159953 + 13464 \\
159953 &= 11 \times 13464 + 11849 \\
13464 &= 1 \times 11849 + 1615 \\
11849 &= 7 \times 1615 + 544 \\
1615 &= 2 \times 544 + 527 \\
544 &= 1 \times 527 + 17 \\
527 &= 31 \times 17 + 0.
\end{aligned}
$$

Hence $\gcd(173417, 159953) = 17$, the last non-zero remainder.

Working back up

$$
\begin{aligned}
17 &= 544 - 1 \times 527 \\
&= 544 - 1 \times (1615 - 2 \times 544) \\
&= 3 \times 544 - 1 \times 1615 \\
&= 3 \times (11849 - 7 \times 1615) - 1 \times 1615 \\
&= 3 \times 11849 - 22 \times 1615 \\
&= 3 \times 11849 - 22 \times (13464 - 1 \times 11849) \\
&= 25 \times 11849 - 22 \times 13464 \\
&= 25 \times (159953 - 11 \times 13464) - 22 \times 13464 \\
&= 25 \times 159953 - 297 \times 13464 \\
&= 25 \times 159953 - 297 \times (173417 - 1 \times 159953) \\
&= 322 \times 159953 - 297 \times 173417.
\end{aligned}
$$